SCIENCE

FRIDAY, MARCH 17, 1899.

CONTENTS:

THE OBJECTIVE PRESENTATION OF HARMONIC MOTION.

CONTENTS.

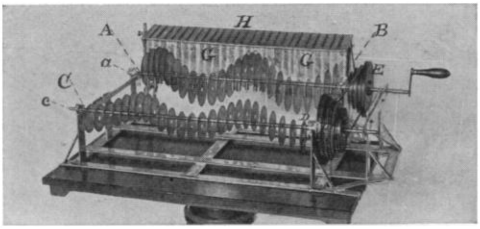


FIG. 1. Belt plate, etc.

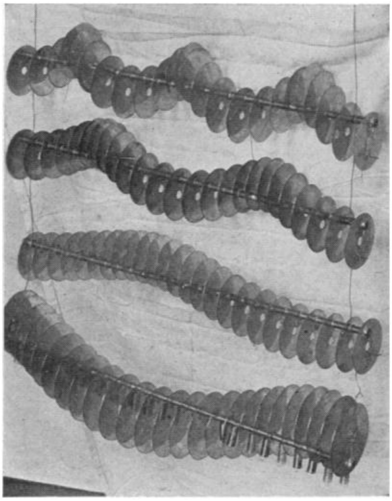


FIG. 2. Cam axles with one, two and three wave-lengths.

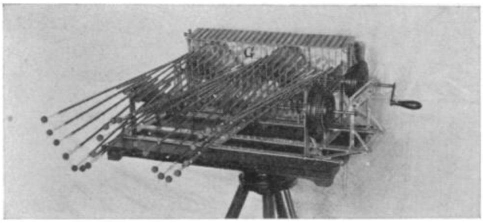


FIG. 3. Adjustment for plane polarized waves.

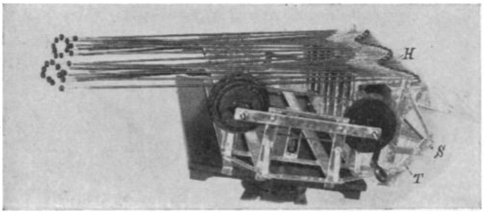


FIG. 4. Adjustment for transverse space waves.

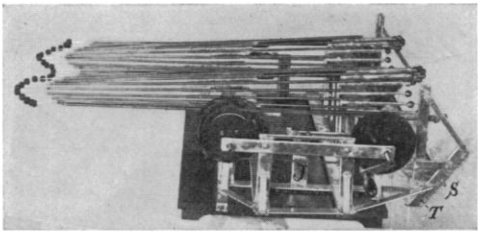


FIG. 5. Adjustment for compounding circular and plane polarized waves.

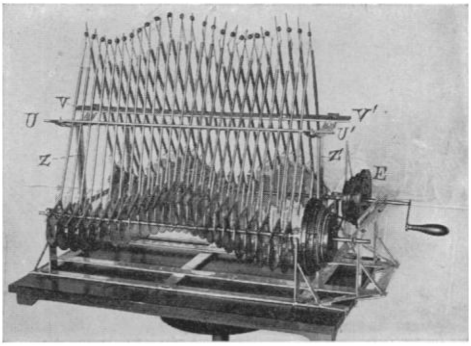


FIG. 6. Adjustment for rotary polarization.

BARUS ON HARMONIC MOTION.

DESCRIPTION OF A WAVE MACHINE.*

1. *Introductory.* — Although wave machines of a variety of special patterns are well known, none of them, to my knowledge, are sufficiently comprehensive in design to embody in a single mechanism the types of harmonic motion met with in acoustics, light, electricity and elsewhere, with a clear bearing on their kinematic analysis. I will, therefore, venture to describe such a machine, even at the risk of becoming prolix, believing the apparatus to be more complete than any similar machine which I have seen, and having, after considerable experience, become assured of its usefulness in class work.

The machine which I have in view must be able, in the first place, to compound any two simple harmonic curves for any difference of amplitude period and phase. The compound harmonic of two, or, at the most, three, components is quite complex enough for illustration, and whatever advantage may be gained from further components is more than counterbalanced by additional complexity of apparatus. The wave machine must next be able to set all the compound harmonics in vigorous motion,* thus producing what I should like to call a train of resolute complex waves (not decrepit waves or waves of deficient vitality); it must do this when the components (meeting at the origin initially in any difference of amplitude period or phase) travel with the same or with different velocities in the same direction or in opposite directions. The latter adjustment affords an admirable illustration of the phenomenon of stationary waves, either with fixed or with wandering nodes; the other an equally apt illustration of musical beats for slight differences of periods or slight differences of wave velocity. Döppler's principle is thus put in evidence. Relative to stationary waves the adjustment is to be set either for reflection with or without change of phase in such a way as to clear up the wretched confusion which usually surrounds this subject in elementary physics.

With these possibilities for plane polar-

*Compiled from notes on lectures delivered at Brown University.

* In this respect the photographs fail utterly to suggest the beauty of the machine when in action.

ized waves, the apparatus must next fully represent the corresponding cases for transverse waves in space. It must, therefore, represent all cases of elliptic and of higher (one might say Lissajous) polarization, both as regards the compounding of harmonic curves for all differences of amplitude period and phase of the two components and the corresponding waves resulting for like or different velocities of the components in the same or in opposite directions. It must show that the section of such waves are Lissajous curves for the particular ratio selected, and that these curves are either fixed or in uniform variation as the component wave-lengths, velocities and periods correspond or not.

The machine should, furthermore, be able to compound simple harmonic and circular motion, showing both the complex harmonics and the waves, to which all variety can be given by changes of amplitude, period and phase. Indeed, types of singular complexity are thus obtainable.

Again, the machine should compound two opposite uniform circular motions, differing in period or wave velocity or both, showing the helical harmonic curves as well as the twisted vibratory waves, with special reference to rotary polarization.

Finally, compressional waves must be obtainable, and this with particular reference to their inherently simple harmonic character.

The machine itself must be made not only of easily replaceable parts and sufficiently simple to resist wear and tear, but so fashioned that the functions of the active appurtenances may be understood from mere inspection. As I have carried it out, the machine is built almost entirely of stout tin plate (about ·027″ thick) folded to secure rigidity, with axles of brass tube to facilitate soldering. Anybody in possession of an ordinary roofman's tin bender* for making lap joints, and a little skill in soldering, can make the machine for himself at a trifling cost.

* The edge around which the plate is bent should be rounded. Sharp bends are not wanted.

2. *General Construction.*—Fig. 1 shows the bed plate of the machine with the attached permanent frame work of tin plate; the movable cam axles, *AB* and *CD;* the driving wheels or pulley cones, *EF*, with belt and crank, and the removable back plates, *GG* vertical and *H* horizontal.

The framework of the rectangular shape seen is made up of strips of tin plate bent into an elongated *C*-section, as shown in Figure 7, firmly soldered together and screwed down to the baseboard. The uprights which carry the axles are similarly made, fastened and suitably braced. A very light and open but strong frame is thus obtained which could be used even without the board. The slight yielding which remains is rather an advantage.

The hollow cam axles *AB* and *CD* of brass, about 25″ long, parallel and 15″ apart at the same height, are sustained at the ends *A* and *C* by pins *a* and *c* secured by metal straps of copper at the ends of the uprights. The pins project about 1″ or 2″ into the axles, so that the latter may revolve around them securely. The ends *B* and *D* of the cam axles similarly receive the reduced and shouldered axles of the pulley cones *E* and *F*, and spring latch pins (one visible at *D*, and in Fig. 2 at the other figures) fasten the pulleys rigidly to the respective cam axles.

Detached cam axles are shown in Figure 2. The pulleys are grooved so that the speed ratios 4 : 2, 4 : 3, 4 : 4, 4 : 6 may be imparted to the axles by successively moving the belt from front to rear. They are mounted in a horizontal rectangle on four uprights corresponding to axles at the corners, and any tension given to the belt bears longitudinally upon the rectangle without straining framework. The rectangle is wide enough to allow the pulleys to slide laterally when-

ever a cam axle is to be removed. This is done for the front axle, for instance, as follows: Let the metal strap at c be loosened and the pin therein withdrawn; this frees the end C. Now let the spring latch at D be withdrawn and the axle of pulley F slid to the right. This frees the end D. The cam axle may now be withdrawn to be replaced by another on reversing these operations.

3. *Cam axles.*—Each of the cam axles (Figs. 1 and 2) carries 25 eccentrics of thick tin plate, equidistant, about 1″ apart and differing in phase by $\frac{1}{12}$ circumference in Fig. 1, so that in this case there are two complete right-handed turns in each of the helices. The diameter of the rear eccentrics is 4″, with a double swing of 3″; the diameter of the front eccentrics is 3″, with a double swing of 2″, but this series has an advantage of position or leverage, as will presently be seen. A safe minimal margin of $\frac{1}{2}$″ beyond the axle is thus left in each case.

It is usually convenient to keep the rear axle in place. In the room of the front axle, however, the other right-handed helices (Fig. 2), containing respectively 1 or 3 turns to the whole length; another containing one right-hand and one left-hand helix (the eccentrics alternating), and a final one left-handed, with 4″ cams and 3″ throw, corresponding to the rear axle (see Fig. 6), are provided. The two latter are adopted for the illustration of rotary polarization. The three former are a means of obtaining wave-length ratios 1:1, 1:2, 2:3 for all amplitudes, periods and phases on removing the front axle only.

The general purposes of the machine will not require more axles than this, though I have used others to be referred to below.

The eccentrics themselves of the heavy tin plate specified are turned together to a common size on the lathe, and soldered to the axle by aid of a suitable gauge. This need merely be a piece of board of a width corresponding to the distance apart of the cams, and having the phase angle carefully marked on both sides. If the board is perforated normally for the reception of the axle, and cut across axially so as to be removable, the soldering of the cam axles is surprisingly easy. I have also tried other methods with success. The work must be done expeditiously, as prolonged heat warps the cams.

The helices shown in the figure are usually right-handed screws. Since they are stationary, a wave advancing from the operator corresponds to counter-clockwise rotation. This is an apparent disadvantage as compared with left-handed stationary screws, but as the waves in the former case advance from left to right (positively for the observer in front) for clockwise rotation by an operator on the right of the machine the disposition chosen is preferable.

4. *Levers, Riders and Balls.*—To obtain the different types of wave motion from the cams described, long extensible levers of thin brass tube are provided, shown in detail in Fig. 8 (longitudinal dimensions $\frac{1}{8}$, cross dimensions $\frac{1}{2}$), and in place in the remaining figures.

The levers were originally made of heavy guttered tin plate behind and light guttered tin plate in front. Latterly, however, I replaced these by the light extensible 'curtain rods' of very thin brass tube,* consisting of a round tube E snugly telescoping into a wider round tube FF, about 5/16″ in diameter. The first tube E is provided with an axial pin R, 3″ long, carrying a 1/2″ cork ball Q (painted red), representing one of the vibrating particles of the wave. The rod R is not seen in sunshine shadows and is added for this reason. Its end is tipped

* These 'rods' are in the market, each about two feet long, thus admitting of a safe extension to much over three feet. Though made of thin split tube, they fit well. The price is trifling.

with an eyelet (not shown), to actuate other apparatus §§8, 25, 24.

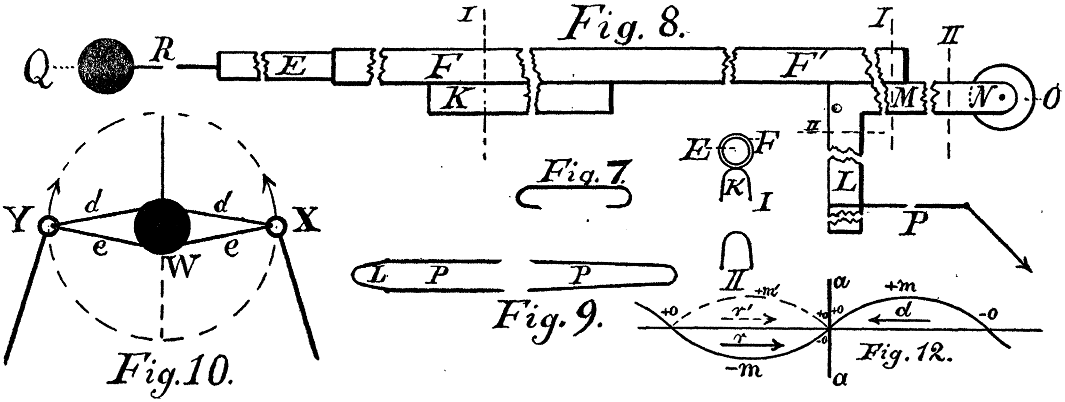
The larger tube FF' carries a U-shaped wide gutter of tin plate K in front, soldered to FF' and adapted to ride on one of the front series of eccentrics. The section through I is seen in the auxiliary figure I, showing all the sectional parts E, F, K in their relative positions. K is $7\frac{1}{2}''$ long and must be carefully placed so as to be adapted to the various types of wave motion. Its position in figure is in the scale specified.

The rear of the tube FF' is fitted with a similar set of gutters of tin plate, M and L, meeting at right angles with their concave sides rearward and downward. This right-angled gutter is adapted to ride on one of the cams of the rear axle, the bearing being on M or L, or both, as the case may be. The section is shown in II. The gutter M terminates in a flat fork, securing and guiding a small vertical roller O, as shown in the figure.

common cross rod (seen at S in Figs. 4 and 5), and they are adjusted as to tension and direction by sliding both ends of the rod S along oblique notched laths of tin plate T on both sides of the machine. The springs themselves appear clearly in Fig. 4, and the riders in different positions in the other figures. The cams in rotating run within the loop of the elastic staple P, and sufficient breadth must be given for clearance. The springs should be as light as practicable to obviate excess of friction on the axle. Steel wire No. 23, wound to a closed helix about 1/2″in diameter and 1/2″ long, is suitable.

The length of the gutter L is 6″, of M to the end of the roller $7\frac{1}{2}''$, and they are soldered toF' to correspond with K.

As regards sure guidance and ease of adjustment, springs placed in the rear of the machine are to be preferred. With less advantage they may be placed between the axles, as was done in my



Figs. 7 to 10. Details.

Fig. 12. Diagram.

To bind the levers firmly down upon the rear cams, a long staple of thin steel wire (No. 16) P is attached about 5″ below FF'. As shown in plan in Fig. 9, this is about 5″ long and pulled downward to the rear by a helical spring the action of which is indicated by the arrow in Fig. 8. The rear ends of all helical springs are soldered to a

earlier apparatus. Levers heavier at their rear ends are desirable, and in some experiments, if not in all, the machine should be tipped up in front. Waves may then be sent along the axis with considerable velocity.

ACTION OF THE MACHINE.

5. *Method of Compounding.*—Very little

need be said about the action of the machine. It is clear that if the rear ends of the levers are horizontally at rest, but execute S. H. M.* in the vertical by riding nearly parallel to themselves on the rotating cams, the balls Q would execute similarly approximate S. H. M. if the fulcrum K were a common axis for all; but if the fulcrum K, though horizontally at rest also executes S. H. M. in the vertical the motion at Q will be the complex harmonic of which the two stated motions (M and K) determine the components.

Again, if the rear end of the lever is vertically at rest, but executes S. H. M. in the horizontal by leaning against the rotating cams rearward (rider L), the ball Q will do the same provided the slide at K is in a parallel plane; but if the rider K simultaneously executes S. H. M. in the vertical the motion at Q is the complex space harmonic corresponding to the two components stated, etc.

Finally the wave-length ratio is given by the cam axles; period ratios are determined by the pulleys or by the velocities of rotation imparted to those axles, respectively; the cam axle and pulley ratios together then determine the velocities of propagation of the component waves.

6. *Plane Transverse Waves.*—With these explanations the remaining figures will be intelligible.

Fig. 3 is the arrangement for plane polarization. In this case all the levers abut at their rear ends against the vertical plate G, with freedom to slide up and down it in virtue of the rollers O (Fig. 8) when the wave is in motion. Grooves for O are an advantage. Riders K and M are here in action, rider L being kept quite in front of the cams by the plate G. The levers are continually pushed to the rear by the clockwise rotation at the crank, and additionally by the rearward action of the springs.

*Simple harmonic motion.

A notched lath (not shown in the figures), stretching quite across the machine between the axles and swung horizontally and upward on a swivel, is adapted to lifting all the levers at once quite above the front cam so as to permit the easy insertion of another cam axle. Riding on this rail the levers show the simple harmonic due to the rear axle alone.

7. *Transverse Space Waves.*—The machine is adjusted for space waves in Fig. 4. Here the rear ends of the levers are lifted so as to roll in the fore and aft grooves of the horizontal plate H in virtue of the rollers O. Riders M are lifted quite above the cams, while riders KL and the grooves on H now control the motion, the levers being drawn rearward by the spring. The figure shows a circularly polarized wave passing along the particles, being compounded of the horizontal rear wave seen on H and the vertical wave above the front axle. Of course, an inspection of the apparatus is more satisfactory.

In a recent construction I have modified the rear plates G and H, discarding H and adopting G in such a way that it may be slid from its vertical position into the horizontal position (H) by following lateral guides much like the platen of a printing press. The plate now carries all the rear ends of the levers with it, which much facilitates the change from plane to space waves and *vice versa.* The grooves on the plate are preferably much wider and deeper than shown in Figure 4.

In Fig. 5 the machine is in the act of compounding the circular motion of the rear axle with the vertical S. H. M. of the front axle. The back plate is wholly removed and the three riders K M L (Fig. 8) now come into play. The figure shows the horizontal S. H. curve, resulting for opposite phases of the vertical components. S. H. structure above the front axle and the circular harmonic arrangement of the rollers in the rear is manifest.

8. *Compressional Space Waves.*—Either of the adjustments, Figs. 4 and 5, is adapted to actuate sound waves, as will be shown below, § 25.

9. *Rotary Polarization.*—Figure 6 shows the apparatus adapted to compound two equal and opposite circular motions, Fig. 10 being a detail relative to it. Both the front and rear series of eccentrics have the same diameter and swing, but there is one turn in front to two in the rear, respectively left and right. The riders are gutters about 4″ long, joined at right angles with the concave sides toward the eccentrics. The extensible levers (tubular as above) are soldered in the prolongation of the bisectrix of the riders, and project from the salient side of the right angle obliquely upward, each passing through a perforation in the horizontal laths of folded tin plate shown at *U U′* and *V V′*. The levers are effectively about 18″ long, and are held down upon the cams by springs * (like the above), one end of each of which engages the lever, while the other is revolubly attached to the axle, between the cams (see Fig. 6). If *U U′* and *V V′* (adjustable) are symmetrically placed with reference to the two effective ends of the levers the upper ends will trace a circle-like figure, corresponding to the circular motion of the lower ends. With the pulleys cross-belted as shown, the pin eyelets *X Y* (3″ long, soldered axially to the upper ends) may then be adjusted to the counter circular motion indicated in Fig. 10.

Two methods of compounding were tried. In the first the ends of two silk threads, Fig. 10, carrying the cork *W* (vibrating particle) between them were fastened to delicate helical springs surrounding the upper ends of the levers. This method constructs the wave very well, but in motion the friction at the eyelets (one of which is often high and the other vertically below it) is apt to be too unequal to keep the particle in the symmetrical position necessary. Better results are obtained by stretching a very thin India rubber band, *dd ee*, between the eyelets, carrying the particle as before. Springs were similarly tested. Parallelogram motion is hardly appreciable here without elaborate construction.

* These springs are seen on the helix in Fig. 2.

The vertical vibration is in this way very well obtained (of course, in semi-amplitude). The horizontal vibration is noticeably curvilinear, seeing that the two motions compounded are not quite uniformly circular. Even in this case, however, the connectors *dd ee* move parallel to themselves.

The helical characters of the wave obtained is well shown in Fig. 6, calling to mind that each ball vibrates normally to the strings by which it is suspended.

The laths *U V* are supported by uprights *Z Z′*, which fit in flat sockets (seen at *J*, in Figs. 4 and 5). With these the whole superstructure of laths, levers and riders is removed from the machine at once in a manner easily suggested. The bed plate then returns to the appearance of Fig. 1.

The method of obtaining similar results in compounding circular motion for the case of Fig. 4 is given below §24. A special cam axle carrying two screws (alternate cams differing 180° in phase) is here needed. If rotary polarization is wanted the wavelengths of the front and rear axle must differ.

EXPERIMENTS.

10. *Method of Designating Phases.*—Before describing the consecutive experiments to be performed with the machine, it is well to come to an understanding as to the phases in which the two component disturbances meet. These are conveniently determined by the *long axes* of the first eccentrics on each axle, which (axes) may, therefore, be called pointers. Since the waves for clockwise rotation at the crank travel from left to right, along the axle, and since a rise of

the front cams elevates the balls, whereas a rise of the rear cams depresses them, the two component waves will meet in the same phase at the origin when the long axes of the eccentrics there point horizontally away from each other, *i. e.*, when the front pointer is to the front or left of the operator at the crank, and the rear pointer to his right. This is also the null position, or zero of phase (to be marked $+0$), for the first particle of each component wave, *i. e.*, the particle on the left hand (origin) of the *observer* facing the machine in front. Both component harmonic curves and the compound harmonic leave the origin with a descending node, the head of the wave (right semi-wave) being a crest. Absence of phase difference in the component harmonics of the particle at the origin will occur for other cardinal positions of the pointers, viz: front and rear pointers respectively up and down (maxima, marked $+m$), right and left or towards each other (mean position marked -0), and down and up (minima, marked $-m$). These follow each other on like clockwise rotation of both axles over 1, 2 and 3 right angles, while the compound harmonic is shoved forward $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ wave-length. Further rotation of 90° restores the original case.

If the two cam axles contain the same number of turns, the same phase difference obviously corresponds to all particles. Otherwise special consideration is necessary for each case.

Restoring the front and rear cam axles to their original positions (pointers horizontally outward), rotate the rear axle 90°, clockwise, relative to the other. This puts all its particles 90° ahead in phase of the corresponding particles of the front wave. The pointers in their cardinal positions will now be respectively left and down, up and left, right and up, down and right to the operator, etc., for successive additional rotations of 90° each.

TABLE OF PHASE DIFFERENCES. CLOCKWISE ROTATION OF AXLES, POSITIVE. DISPLACEMENTS OF PARTICLES UPWARD AND REARWARD, POSITIVE.

Transverse Plane Waves.	Transverse Space Waves.	Front Axle.	Rear Axle.	Front Axle.	Rear Axle.	Front Axle.	Rear Axle.	Front Axle.	Rear Axle.
Same Phases.		+0	+0	+m	+m	−0	−0	−m	−m
		←	→	↑	↓	→	←	↓	↑
	Front Axle } + 270°, Rear Axle } + 90°	+0	+m	+m	−0	−0	−m	−m	+0
Front Axle } + 90°, Rear Axle } +270°		+0	−m	+m	+0	−0	+m	−m	−0
		←	↑	↑	→	→	↓	↓	←
	Same Phases.	+0	+0	+m	+m	−0	−0	−m	−m
Opposite Phases.		+0	−0	+m	−m	−0	+0	−m	+m
		←	←	↑	↑	→	→	↓	↓
	Front Axle } + 90°, Rear Axle } +270°	+0	−m	+m	+0	−0	+m	−m	−0
Front Axle } + 270°, Rear Axle } + 90°		+0	+m	+m	−0	−0	−m	−m	+0
		←	↓	↑	←	→	↑	↓	→
	Opposite Phases.	+0	−0	+m	−m	−0	+0	−m	+m

Beginning again with pointers away from each other, *i. e.*, with both component, S. H. motions starting at the first particle, let the front axle be rotated clockwise 90° relatively to the other. The pointers in their cardinal positions will now be up and right,

right and down, down and left, left and up, while the particles at the origin run through all phases together. This case corresponds to the preceding for 270°, etc.

All this is evident enough; but it is, nevertheless, advisable to make a diagram of the position of the pointers as here shown, in order instantly to discern the phases in which the initial particles meet in any case. In the table the positions of the pointers are designated by arrows; $+m$ denotes maximum displacement, etc. Further explanation will be given presently.

11. *Space Waves.*—The composition of two simple harmonics at right angles to each other will necessarily require special treatment, for here the rear riders are at right angles to those of the former case, and the S. H. motion of the rear axle is not reversed at the balls. If displacement up and forward from the observer's view be considered positive, then the null position or zero of phase of the particles at the origin corresponds to pointers, left for the front axle and up for the rear axle, as seen by the operator. The compound simple harmonic of these components is thus a linear vibration with amplitude $\sqrt{2}$, as regards the equal components, and making an angle of 45° to the horizontal from the observer to the machine. It thus lies in the first quadrant, as seen by the operator at the crank.

Both component S. H. curves leave the origin with a descending node.

Hence, if in the above table we shove the first column of entries one row ahead, *i. e.*, if we begin for no phase difference with the second row and continue in cyclical order, the table will be adapted to the present case. Pointers in opposite directions will thus correspond to counter-clockwise circular motion in the compound wave; pointers in the same direction to clockwise circular motion, as seen by the operator at the crank. The first of these cases will, however, correspond to a right-handed, the second to a left-handed, screw when seen from the origin, since all waves move from left to right.

The table contains an entry relative to the present case. It thus indicates 16 cardinal phase differences for plane and the same number for space waves.

12. *Effective Circles of Reference.*—Finally, a word may be said as to the position of the circles of reference corresponding to the two component S. H. motions. Clearly, the centers of the eccentrics (marked in Fig. 1) determine the amplitude of the S. H. M. In all phases, however, the riders are nearly normally above or else to the rear of these centers by a distance equal to the radius of the eccentric, and, therefore, always in the same kind of reciprocating motion which corresponds to the amplitude and period of the eccentric.

Hence the circle of reference of the vertical S. H. M. is on a vertical diameter and tangent to the highest and lowest positions of the edge of the eccentric on the same side of the axle. The diameter prolonged passes vertically through the cam axis, and its length is twice the throw of the center of eccentric. This circle of reference for the horizontal S. H. M. of the riders (displacement + rearward) is on a horizontal diameter and tangent to the extreme right and left positions of the edge of the eccentric on the same side of the axle.

The amplitude of the vertical vibrations is modified by the lengths given to the extensible levers. If l be the lever length between the axles and l' that beyond the axles, and if a, a' denote the front and rear amplitude at the eccentrics, then the effective amplitude at the particles will be $a(l+l')/l$ and $a'l'/l$, and their ratio

$$\frac{a'}{a}\,\frac{l'}{l+l'}$$

may be varied at pleasure from zero to about 9/8, since l' is the extensible part. Usually the ratio one is desirable.

The amplitudes of the horizontal vibrations do not admit of change without giving useless complexity to the machine. Advantageous lever ratios will be given with the experiments.

I. *Component S. H. Motions Coplanar, of the same Wave-Length.* 13. *Plane Polarization.*—Let cam axles each with two complete turns be selected and the rear plate adjusted to the vertical (Fig. 3). For harmonic curves this implies the same wave-length for the coexisting S. H. motions. With the cams swinging nearly as 2 : 3, and lever ratios $l + l'$ and l' (§ 12) as 3 : 2, the occurrence of no displacement along the line of particles may be looked for in case of opposite phases. This furnishes a method of adjusting the particles at the outset. Practically the condition of no displacement is reached with relatively short levers, say a meter long. When the pointers on the initial cams are away from each other the components meet in the same phase, with the first particle in the axis of motion just about to start vibrating. The double amplitude given by the machine to this compound harmonic (25″ long) of maximum displacement is about 9″. If a beam of parallel rays (sun light) be shot along the axis of the wave, the shadow of the balls on a screen normal to the axis necessarily betrays slight curvature; the double amplitude, instead of being vertical and straight, is concave toward the cams. But the chord deviates from the arc (9″) by less than 1/2″ at the center, and hence with balls 1/2″ in diameter the curvature is negligible to the eye of an observer in front. It must be remembered, however, that curvature is superimposed in all subsequent higher figures.

If the front cam axle be dephased 90° clockwise the amplitude of the compound curve is diminished, the curve remaining sinusoidal but beginning with 1/8 wave-length. If the rear cam axle is also dephased 90° clockwise the compound curve of the first case is restored in the shadow (maximum amplitude), but the phase of the first particle has advanced 1/4 period and the curve itself 1/4 wave-length, etc. I allude to these points because of their value in instruction. (Cf. table, § 10.) By the very make-up of the machine a S. H. curve is seen to result when the phase difference of two particles varies as their distance apart. In drawing such a curve it is simpler to place the circle of reference in the plane of the harmonic; in the machine the circle of reference is preferably placed at right angles to the curve. The addition of two such curves is another S. H. curve of the phase and amplitude directly specified by the machine.

14. *Waves of Constant Amplitude.*—Belting the two equal pulleys and rotating uniformly, waves corresponding to each of the harmonic curves produced in §13 may be sent along the axis of motion. Thus making the phase difference between two particles proportional to their distance apart, and then setting each particle in S. H. M. of a common period and amplitude, is objectively seen to be the realization of simple wave motion. The wave-length being fixed by the apparatus, velocity and period must vary reciprocally.

Particularly striking is the case for opposite phases in the two wave cams. Both component waves are seen travelling in the same direction along the axes with full vigor, whereas the compound effect at the line of particles is permanently nil.

The warped surface of the levers now has a linear directrix at the particle edge and a sinusoidal directrix at the roller edge. It should be noted that the case of maximum amplitude in the compound harmonic presents an approach to a similar linear directrix between the cam axles.

15. *Waves of Varying Amplitude.*—Change of amplitude is given to the levers by draw-

ing out the front tube (§12). Additional change may be obtained by allowing colored balls to ride on the levers. In case of equal periods the result is chiefly interesting when the amplitude varies from particle to particle. A linear variation is well represented by a plane wave oblique to the direction of the axles, and in action is very striking.

The more important wave with an exponentially varying amplitude is only given when the axis of motion is along the corresponding exponential curve horizontally, but the effect to an observer at a little distance in front is none the less good.

II. *Preceding Case (I) with Additional Velocity Superimposed on Either Wave Train.* 16. *Beats.*—If the component waves are transmitted in like periods or velocities* and amplitudes, the compound wave is transmitted unchanged in form; but if any of these quantities vary, the compound wave continually changes form. With the apparatus as here adjusted the last case is readily realized by sending on one wave faster than the other. For instance, if the component wave velocities be as 3 : 4 (rear wave of greater speed), then in 4 complete turns at the crank the original wave will be reproduced, while all intermediate phase differences between corresponding particles are passed continuously in turn. All pairs of cams are undergoing like continuous change of phase.

The shadow picture of this case (sunlight) shows a line elongating to maximum displacement and then contracting to a point in S. H. M. The slow change at maximum elongation is in strong contrast to the rapid change of length on passing through the position of equilibrium. Similarly in §14 the speed ratio must be carefully adjusted if the linear compound wave is to persist.

* In the present special case variation of one implies the other; in the sequel, period and velocity must be carefully distinguished.

The wave corresponding to this present experiment is an excellent example of an infinite beating wave train, two wave-lengths of which are accessible at a given place. The beats are due to a difference of wave velocity and frequency together. Though the two cases are usually generically different, the gross effect is here coincident. As a luxury a cam axle containing a small fraction of a wave-length more than two complete wave-lengths might be supplied. This would then show beats due to difference of wave velocity for the same period or (with the proper pulley) beats due to difference of period for the same wave velocity. The specific difference is this, that, whereas in one case (equal component wave-lengths) the compound harmonic is at every instant (for all pulley ratios) sinusoidal, in the other case (slightly different component wave-lengths) it is at no instant strictly so. The latter adjustment thus admits of beats either when the component periods alone or the component wave velocities alone are not the same. In the former both necessarily change together.

17. *Döppler's Principle.*—If the beats are obtained by a difference of wave velocity the faster wave may be treated as having an additional linear velocity *virtually* impressed upon it in the direction of motion from without. Its interference with the wave not so affected is then an illustration of Döppler's principle.

III. *Preceding Cases (I and II) with the Velocity of Either Wave Train Reversed.* 18. *Equal Velocities. Stationary Waves.*—If one of the component waves be passed along the axis positively and the other in a negative direction, *i. e.*, if one axle be rotated clockwise and the other counter clockwise by cross-belting equal pulleys, the compound wave is of the stationary type, since amplitudes were made effectively equal and periods are necessarily equal. The effect on the machine is striking, since the nodes are

here indicated by stationary particles half a wave-length apart, while the antinodes vibrate 9″. In all positions the form of the compound harmonic curve is at all times a simple sinusoid, but its mode of motion as compared with the same curve while both components are direct is totally different.

Again, if the first pair of cams are in the same null phase (pointers away from each other) the first particle is a node, succeeded by four other nodes one-half wave-length apart, and the wave is initially at maximum amplitude. If the first pair of cams are in opposite null phases ($\pm O$) the initial harmonic curve is linear, the first of four nodes one-fourth wave-length ahead, etc.

Reflection.—The first of these cases corresponds to reflection from a denser, the second to reflection from a rarer, medium at the origin. It is worth while to examine the interpretation of both cases* for transverse waves first, and thereafter, §26, to similarly treat longitudinal waves.

If the direction of a wave is reversed, particles without displacement ($\pm O$) are changed half a period in phase (becoming $\mp O$); particles at maxima or minima ($\pm m$) are not changed in phase at all, while the phases of intermediate particles are changed in the corresponding harmonic ratio. This may be tested at once by supposing the full wave, Fig. 12, to advance first in direction d, thereafter in direction r, when the particles vibrating in the line aa will respectively rise and fall, thus passing between opposed phases; etc.

The transverse wave advances through a given medium at rest, with the zero of displacement ($\pm O$) in the wave front, so understood. Hence to reverse the direction of a wave is to reverse the phase of the wave front.

If the transverse wave encounters a denser medium this implies that the particles therein situated are capable of *reacting with forces in excess* of those corresponding to the original medium. If the medium is quite impermeable (as when the wave on an elastic cord meets the peg) the reaction is exactly equal and opposite to the action. Thus if a wave advances toward the dense medium with a crest or group of pulls upward the medium itself must at every instant react with equal pulls downward. This reaction, which in its succession is bound to be rythmic like the impinging wave, is the impulse of the reflected wave, which must all be returned into the first medium (*i. e.*, be reversed in direction) if none can enter the new medium.

Now, let the particle in the wall aa (Fig. 12) be in the zero of phase ($+O$). The direct wave advancing, as shown by d, is in the act of increasing the displacement. It is developing an increasing pull up. The reflected wave (prolonged) r is simultaneously in the act of developing the counter pull down; it is, in like degree, tending to decrease displacement: but, though the phases impressed by the direct and reflected wave are thus initially quite opposite, both waves d and r momentarily constitute contiguous parts of the same harmonic curve. If this curve separates at aa, with the parts d and r moving with equal velocity in opposite directions the condition for action equilibrated by reaction at aa is maintained throughout all time.

The explanation is essentially the same if the reaction is not complete (permeable dense medium). In this case the amplitude of r will be smaller, other conditions remaining the same.

Hence in the machine the pointers are to be set for equal and opposite displacements at the origin, beginning with the null phases of each component wave—the case of Fig. 12, where if d and r were moving in the same direction, or the pulleys not cross-belted, the two components would meet in the same place.

*Waves as here considered are essentially steady.

On the other hand, if the direct wave meets a rarer medium at *aa* the reaction is less than that of the original medium. The pull up developed by the wave *d* in Fig. 12 is not resisted by an excessive pull down as before. The reaction (which from its rhythmic character develops the reflected wave) is an additional pull up, such as would correspond to a wave *r′* in Fig. 12. Both waves *r′* and *d* are in the same phase as regards their effect on the initial particle at *aa*, but they differ in direction of motion. In other words, the direct wave *d* and the reflected wave prolonged *r′*, are not initially contiguous parts of one and the same wave, meeting without displacement at the wall. Half a wave-length is necessarily lost at the inception.

This determines the method of setting the pointers of the machine for equal displacements of the same sign at the origin, beginning with opposite null displacements; for the two waves *d* and *r′* if traveling in the same direction (cf. Fig. 12) would then annul each other.

Summarizing; the reflected wave from a denser plane boundary normal to the axis is obtained from the incident wave by *two* rotations of 180° each; one around the axis of motion, the other around the trace of the wave plane on the plane of the obstacle; these correspond respectively to the substitution of reaction for action, and of an opposed direction for the given direction of motion—two reasons for change of phase. The wave advancing *crest on* (crest foremost) returns *trough on* and *vice versa.*

The reflected wave from a rarer plane boundary is obtained from the incident wave by a *single* rotation around the trace in question. The only reason for change of phase is change of direction. The wave advancing crest on returns crest on, and the trough returns a trough. Cf. §26.

If the component amplitudes are made unequal the nodes show a correspondingly slight vibration, the case corresponding to a medium at the origin neither absolutely impermeable nor absolutely rare.

19. *Wandering Nodes.*—If with equal amplitudes the velocities or periods of the components be unequal in value and opposite in sign the case becomes one of stationary waves with continually drifting nodes. Thus if the 3 : 4 pulley be cross-belted four turns of the rear or faster cam axle will continuously move the node half a wave-length onward. The stationary character is, nevertheless, very thoroughly retained.

In the extreme and transitional case where the velocity of one wave is zero and the other of any value a single turn at the crank moves the nodes half a wave-length and thus reproduces the original curve.

IV. *Component S. H. Motions at Right Angles to Each Other of the Same Amplitude and Wave-Length.* 20. *Elliptic Polarization.*—Using cam axles with two waves each and adjusting rear ends of levers (Fig. 4), while the vertical riders *L* engage the cams, two simple harmonic curves are available to be compounded at the particles. This is usually an elliptic helix. It is advisable to tip the machine up in front with the object both of relieving the work of the springs and of exhibiting the wave symmetrically with reference to a horizontal plane through the axis.

In order that circular polarization may be obtained, the amplitudes of the particles must be equal. The rear cams contribute their full swing independent of the levers. The fore cams enter with an amplitude which may be more than doubled, though the fulcrum of the levers is now at the rollers. Thus the levers are to be shortened from 1 meter to about 70 cm. to obtain circular paths 3″ in diameter for the single particles. Shorter levers would give oblate ellipses, larger levers prolate ellipses, for their central figures. Cf. §36.

The two S. H. motions will meet and

exist throughout in the same phase if the pointer on the rear eccentric is 90° ahead of the other, supposing, in accordance with the above table, that directions upward and rearward are positive. The zero of phase thus begins with front pointer left and rear pointer up. If the pointers are parallel and in the same direction the front harmonic is 90° in phase ahead of the other. The compound harmonic is circularly polarized and the corresponding wave advances with counter-clockwise rotation if seen in the direction of advance, *i. e.*, from left to right to the observer in front. Dephasing the front axle 90° farther (180° advance) produces plane polarization at 135° to the horizontal; 90° farther (total advance 270°) finally a circularly polarized harmonic curve with a wave advancing in the direction of the components with clockwise rotation, as seen from the origin. All intermediate cases are elliptically polarized with intermediate rotation.

The sunshine picture on a screen normal to the axis with rays parallel thereto is in general an ellipse with the appropriate rotation discernible with remarkable clearness.

V. Preceding Case (IV) with Component Velocities or Periods Unequal. 21.—If the component waves do not advance with the same velocity (necessarily implying difference of period in the present case) the difference of phase of the first pairs of cams is continually changing, and the phase difference of all succeeding cams is changed in like measure. Hence the compound wave passes continuously through all the different harmonic curves in turn. If the belt be placed on the 3 : 4 pulleys four turns of the rear axle restores the original form through all intermediate forms, beginning, for instance, with plane polarization at 45°, passing through circular clockwise polarization (seen from the origin) into plane polarization at 135° ; then back with counter-clockwise rotation into plane polarization at 45°.

The sunshine shadow of this case is identical with the Lissajous figures from two tuning forks slightly different in pitch but of the same amplitude. The directions of rotation are particularly evident, enhancing the instructivenes of the figure.

VI. Preceding Case (IV.) with Either Component Velocity Reversed. 22.—If with equal amplitudes and wave-lengths the component waves travel in opposite directions (pulleys cross-belted) the compound wave is a peculiar form of stationary wave in which the form of vibration of all particles is sustained, but in which the motion of each particle differs uniformly in regard to the phase difference of its components, *i. e.*, in ellipticity, from its neighbors. Thus a group of particles a wave-length apart are plane polarized at 45° ; particles midway between plane polarized at 135° ; particles midway between both groups circularly polarized with alternately opposite rotations and all other particles correspondingly elliptically polarized. The envelope of the harmonic curve would be given by a thin tube 3″ in diameter, compressed at equal distances by a pair of shears to lines at right angles to each other, but alternately in the same direction. The case is thus thoroughly different from the case of unequal velocities in the same direction, where all the particles under observation are instantaneously in the same ellipticity.

23. *Velocities Reversed and Unequal.*—If the two component waves of the same wave-length have unequal velocities (and periods) of opposite sign the plane polarized groups wander. Thus if the 3 : 4 pulleys be taken 4 turns of the rear crank reproduces the original wave. The transitional case is again that in which one cam axle is stationary (wave velocity zero) while the other rotates. A single turn reproduces the original figure.

VII. *Preceding Case (IV.) Adjusted for Rotary Polarization.* 24.—If a special axle be provided with the cams alternately in opposite phase to the normal occurrence, but otherwise equal in amplitude and wavelength, and if the corresponding balls be painted red and white, the two circularly polarized waves occur simultaneously. Similarly, the two plane polarizations at 45° and at 135° occur simultaneously; etc. The former case is interesting in relation to rotary polarization, as will be more fully indicated below; for the two circular motions may be compounded by the device shown in Fig. 10, and a harmonic curve plane polarized in the vertical or the corresponding wave will result (cf. §40 *et seq.*).

To obtain rotation of the plane of polarization by this method the alternate cams on both the front and rear axle would have to be set for some other wave-length in the manner stated.

(vertical) axles of which are at the angles of the cranks, as far apart as the cams, and all arranged along a straight line parallel to the cam axle. The short shanks of the bell cranks now carry a series of $\frac{1}{2}''$ balls, which, under present conditions, must, therefore, vibrate nearly parallel to the cam axles, *i. e.*, longitudinally right and left in the line of advance of the wave, whereas the thrust of the levers* is harmonically to and fro.

In practice the long shanks are open sectors of wire, swung so as to clear each other's axles.

In this way the alternate compression and rarefactions of such a wave are remarkably well shown (cf. Fig. 11), the sinuosity in the line of particles being negligible at least to the observer in front. The balls approach each other to about $5/8''$ between centers (all but contact in the compressional phases), while they sepa-

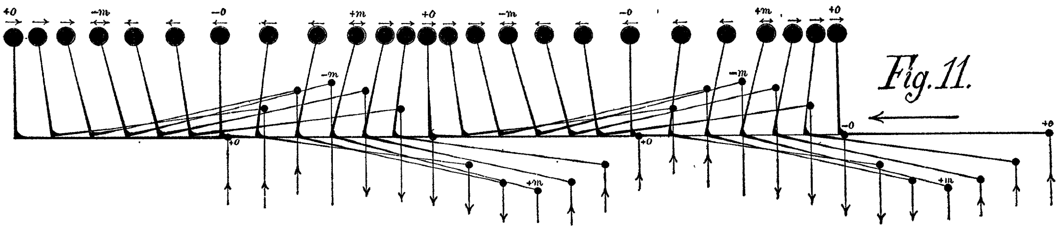


FIG. 11. Adjustment for compressional waves, seen from above. Diagram. Wave advances from left to right to an observer in front. Lever displacements positive rearward. Ball displacements necessarily reversed.

VIII. *Waves of Compression and Refraction.* 25. *Longitudinal Vibration.*—With the apparatus arranged as in Fig. 4, let the levers all be raised at the front ends, so as quite to disengage them from the front cam axle. This being, therefore, out of action, the rear or horizontal harmonic of $3''$ double amplitude forward and rearward thrust is alone in play, as shown in plan by the parallel lines normal to the axis in Fig. 11. Now, let the ball ends of the levers (eylets) engage the long shanks ($6''$) of a series of horizontal, *right-angled* bell cranks, the equidistant

rate to more than about $1\frac{3}{8}''$ in the rarefied phases.

The great advantage of an arrangement of this kind from the kinetic point of view is the direct evidence furnished that each ball in the first instance is actually in S. H. M., and that the phase difference between balls is proportional to their distance

* The reader should remember that Fig. 11 is seen from above and that direction rearward in the transverse harmonic (down in figure) is positive wave velocity left to right in the machine, becomes right to left in figure. Balls in front reverse their positive motion.

apart, while the compression and rarefaction of such a wave is an incidental phenomenon. This essential structural character of the acoustic wave is not generally enough insisted on.

In the same way any of the above or the following complex plane polarized waves may be converted into compressional waves by using vertical bell cranks (horizontal axes). For the case of stationary waves this would be of some interest, but I have not carried out the construction. Since small displacements are wanted, the engagement of levers should be located between the axles.

26. *Reflection.*—There is bound to be confusion if the reflection of a compressional wave from a denser or a rarer medium is to be explained without reference to the elementary S. H. M. of the particles of such a wave. Relatively to § 12 and Fig. 12 it follows that the explanation there given is at once applicable to S. H. M. in sound waves, the only difference being that for pulls up and reactions down we have now pulls toward the right and reactions toward the left, etc., which in no way modifies the reasoning. A wave advancing toward the dense medium 'crest on' returns 'trough on'; advancing toward the rare medium with a crest returns a crest. Let no one suppose, however, that crest and trough mean compression and rarefaction. For it is just here that a slough of despond awaits the incautious interpreter. A glance at Fig. 11, where the oscillations of particles have all been marked, shows that *the centers of compression and of rarefaction are without simple harmonic displacement (phases* ± 0*); that the maxima and minima of displacement (*$\pm m$*) lie in air of normal density.* If the wave is to advance with particles in the wave front in the zero of displacement it must advance the center of a compression or the center of a rarefaction sharply into normal air. Thus, the particles on one side only of the balls marked ± 0 in Fig. 11 must indicate the status of an advancing sound wave; moreover, if the former begins a crest, the latter (particles on the other side of ± 0) begin a trough, and *vice versa.*

In this structural fact lies the gist of the true explanation: If a compression meets a denser medium it is reflected as a compression surely enough, but the two compressions are not the same. The symmetrical half of the incident compression is returned. The two halves lie on opposite sides of no displacement, and are the contiguous halves of crest and trough required by Fig. 12. So the two symmetrical halves of a rarefaction become incident and reflected wave, initially meeting the plane of reflection as contiguous trough and crest. In both cases crest returns trough, and trough crest, even though two compressions or two rarefactions are in question.

If reflection takes place from a rarer medium a compression returns a rarefaction; this, however, is the rarefaction ending in a crest, while the given compression begins one, and *vice versa.* In other words, there are two crests advancing in opposite directions; or crest returns crest, even though a half wave-length is initially lost and though a compression returns a rarefaction.

The agreement with §12 is thus complete and the whole explanation logically simple throughout.

IX. *Component Simple Harmonics Coplanar, with Wave-Length Ratio, 1 : 2. Harmonic Curves.* 27.—Replacing the front cam axle by another containing a single wave-length and 2″ double amplitude, the plane compound harmonics for period ratio 1 : 2, for the same or different amplitudes and for any difference of phase, may be exhibited in succession. The cams are exchanged by lifting all the levers above the front axle, by aid of the notched swivelled cross-lath

(when an opportunity to show the rear harmonic *alone* is afforded as the levers now ride on a common fixed axle in front), after which the single wave axle is easily inserted and the levers dropped down upon it by lowering the cross-lath.

Reference to the scheme of phases compiled in §10 shows that 16 generically distinct compound harmonics with an indefinite number of intermediate curves are obtainable. The variation is further enhanced by changing the component amplitudes by drawing out the levers. Among forms for equal amplitude the symmetric types are distinctive. They are obtained concave upward more or less *W*-shaped for components meeting at the origin both at maximum displacement ($+m$), and more or less *M*-shaped when both components meet at the origin at minimum displacement ($-m$). Similarly symmetrical forms are seen when the components at the origin are in opposite phases, viz., *V*-shaped when the front harmonic is at $+m$ and the rear harmonic at $-m$, and Λ-shaped when the front harmonic is at $-m$ and the rear at $+m$.

28. *Waves.* If these curves are to be transmitted in a compound wave which does not change its form each component must travel equally fast. Hence the rear axle with two wave-lengths must be rotated twice as fast as the front axle with one wave-length (pulleys 2 : 1) The periods are now also in the ratio of 1 : 2. Thus it appears, that it takes two rotations of the rear axle to exhibit the complete wave, or beginning with a symmetric type, for instance, the *W* and Λ curve together make a single harmonic curve; whereas the *M* and *V* curve make another, in relation to waves; etc., for non-symmetrical forms. The character of the wave is markedly progressive, each little kink as well as large elevations or depressions running along the axis in turn.

Referring again to the above table §10, the present succession of phases is a march along a *diagonal* passing from left to right downward across the diagram.

29. *Case IX. with Component Velocities Unequal.*—If the component waves are transmitted unequally fast the compound wave continually changes form. Thus, if the 2 : 3 pulleys be used, it takes 3 turns of the rear axle to reproduce the original form ; in 3 : 4 pulleys, four turns ; in 1 : 1 pulleys, but a single turn. In the last instance the waves produced are much like stationary waves, with two nodes at the ends if the components meet at the origin in opposite phases, and one node in the middle if they meet in the same phase, phase difference being maintained constant at each cam. The table, §10, shows that the passage is now from left to right across the diagram, along a single row.

If one axle alone rotates a single turn again reproduces the original form, but the wave has now a progressive character, which is an inversion of the result in §28. In other words, the *W* and *V* types or the *M* and Λ types of curve are successive. In the table of phases, §10, the present succession for any single cam is given by a column passed from top to bottom.

30. *Case IX. with One Component Velocity Reversed.*—If the axles rotate with equal velocity in opposite directions the wave presents the succession of forms of the first (normal) case, but its character is now non-progressive, each particle retaining its peculiar form of vibration, which differs regularly from that of neighboring particles. But half the full wave is represented at once. No particle is permanently at rest and the stationary character is less pronounced than for the case in §29 with equal pulleys. Particles at the end of the curve in view are in like figures of vibration. In the above table, §10, the passage for any single pair of cams is now diagonally across the diagram, but from right to left, downward.

X. *Components Simple Harmonics at Right Angles to Each Other, with Wave-Length Ratio, 1:2. Transverse Space Wave.* 31. *Harmonic Curves.*—With the preceding cam axles, let the rear ends of the leaves be lifted upon the horizontal back plate and adjusted for the same component amplitude (Fig. 4).

Space waves of this and the following kind may be conveniently termed Lissajous waves, since their sunshine shadow on a screen normal to the axis of motion is always the appropriate Lissajous figure. Starting the waves with the initial eccentrics towards each other, the harmonic curve has a meandering space form, characterized, however, by its sunshine shadow, which is the specific bow-shaped 1:2 Lissajous, concave toward the cams. Dephasing the rear axle $+90°$ produces the symmetrical 8-shaped figure; $+90°$ farther the bow-shape again, this time, however, convex toward the axles of the cams; $+90°$ farther returns the 8-shape described in a direction opposite to the preceding. The intermediate cases are assymmetrical 8's, but not well given unless the balls are small enough.

The harmonic curves themselves present no marked complexity. Seen from above they contain two wave-lengths; seen from the front but one wave, each in the appropriate phase at the origin. This gives a very clear analysis of the occurrences. The wave envelope in the bow-shaped cases is a gutter.

32. *Waves.*—The waves corresponding to the above space harmonics are instructive. If the figure of the compound wave is to be preserved, *i. e.*, if its shadow Lissajous is to remain fixed, both component waves must advance with rigorously the same velocity. This implies double rotation (double frequency) for the rear waves of shorter wave-length. The direction of rotation in the shadow is particularly well marked. For initially opposite or for like phases at the origin the figure is alike 8-shaped, but when horizontal pointers on the front axle correspond to down on the rear or up on the rear the rotation is clockwise or counter-clockwise respectively in its upper half; etc.

33. *Case X. with Component Velocities Unequal.*—If the velocities of the component waves are unequal, but of the same sign (pulley 2:3, for instance), the compound wave continually changes form, as is best shown by the sunshine shadow. This is identical with the Lissajous curve for two tuning forks of the same amplitude, but with period ratios slightly different from 1:2. If the speeds of the two axles are equal (pulleys 1:1) a single rotation of the crank produces all the Lissajous between two occurrences of the same figure.

If the component periods are equal, but of opposite sign, stationary wave conditions appear for this case. Particles at the ends of the compound wave oscillate in any fixed Lissajous; the intermediate particle has the inverse figure. In general the permanent vibration figures vary proportionally to the distance apart of the particles. The sunshine figure is reproduced for 1/2 rotation at the crank. One may note the contrast that, whereas the particles themselves vibrate in the elliptical Lissajous series, the sunshine shadow produces the 2:1 series.

If the component periods are unequal and opposite in sign the figures drift as above. The transitional case is given when but one axle rotates.

XI. *Component S. H. Motions Coplanar with Wave-Length Ratio, 2:3.* 34. *Plane Harmonics and Waves.*—The front cam axle is replaced by one containing 3 wave-lengths, with adjustments as above (Fig. 3). The curves of this series are more complex than the preceding, and if the dephasing be effected in steps of 90° each, 16 marked forms of curves may be exhibited. Among these the symmetrical types are

best adapted for recognition. They correspond respectively to like phases at the origin with maximum or miminum displacement of both components (*W*- and *M*-shaped forms), or to opposite phases at the origin with maximum and minimum, minimum and maximum displacements of the components (*V*- and Λ-shaped forms).

If the component waves are to advance with the same velocity the rear cam axle rotates twice while the fore axle rotates thrice, thus establishing a period ratio of 3 : 2. Hence each wave contains two of the specified harmonic curves in succession, or only one-half of it is seen at once. The progressive character of these waves as they dash along is singularly pronounced.

If the axles rotate equally fast in the same direction the wave assumes a stationary type, with one node at the middle of the component harmonics meeting at the origin in the same phase. If the latter meet at the origin in opposite phases, nodes occur at the two ends with marked vibration for intermediate parts of the compound wave. If the cam axles rotate equally fast, but in opposite directions, the compound wave shows 6 nodes if the components meet in opposite phases at the origin, and 5 nodes under other conditions.

Finally, if the wave velocities are equal, but opposite in sign, there is permanence in the vibration form of each particle, with difference of phase between them, but no nodes.

XII. *Component Simple Harmonics at Right Angles to Each Other, with Wave-Length Ratio 2:3*. 35. *Transverse Space Waves.*—The results are similar to the above cases, only more complex. The sunshine shadow on the normal screen shows the 2 : 3 Lissajous figure in permanent form if the axes are rotated at angular velocities of 3 : 2. The component waves are then transmitted with equal velocity and the period ratio becomes 2: 3. If the component waves are transmitted with other velocities the compound wave continually changes form, as does also the Lissajous shadow curve. The rotation within it is here again exhibited as to direction, etc., with remarkable clearness. To obtain steady results for this case the balls must be small and the ratio workmanship of the machine accurate, otherwise the inconmensurable cases supervene. Experiments are made as above.

XIII. *Component Harmonics Circular and Vertically Simple Harmonic of any Wave-Length Ratio*. 36. *Harmonic Curves for Equal Component Wave-Lengths.*—The present curves are interesting, inasmuch as they present an intermediate stage between the above cases of S. H. composition and the next cases relating to the composition of circular motions. The wave machine is put into adjustment, as shown in Fig. 5, with cam axles and pulley ratios 1: 1. The machine is tipped up in front.

Inasmuch as the S. H. M. of the front axle interferes with the vertical component of the circular motion of the rear axle, the phase difference is best specified in terms of these coplanar vibrations. For like phases, therefore, the Lissajous figure of the compound curve is a tall vertical ellipse, say 9″ high and 3″ broad. Advancing the front phase +90° inclines this ellipse to the rear, shrinking it throughout. Advancing the front axle +90° farther produces the simple harmonic curve in the horizontal with a double amplitude of 3″. The further advance of the front phase of +90° expands the Lissajous figure into an oblique ellipse inclining to the front, etc.

37. *Waves.*—The rotation in the waves is always clockwise for a clockwise circular component. In this and other respects (pronounced prolateness combined with horizontal plane polarization) they differ from §20.

38. *Waves and Curves for Other Component Wave-Lengths.*—On replacing the front cam axle with one of one or three waves to the

two of the rear axle, peculiar apparently beknotted wave forms are obtained, well adapted to give a notion of the complexity resulting from simple compounding; but it is needless to refer to them further.

XIV. *Component Harmonics Both Circular, of any Wave-Length Ratio and Opposite in Direction.* 39. *Remarks on the Machine.*—After the description of the machine and the remarks already made in the successive paragraphs above, it is not necessary to enter at length into a consideration of the present experiments. As to matters of adjustment in Fig. 6, I may note that the common horizontal locus of the centers of the approximate circles described by the free ends of the levers (they are really curves of the 6th degree), and the respective cam axles, must be equidistant from the perforated cross laths, *U* and *V*. In the given apparatus the effective lever length is about 18″. In this case the lever ends describe curves which do not differ more than 1/8″ from circular circumference, a departure not discernible with 1/2″ balls. Nevertheless, the angular velocity in the quasi-circles is not uniform, a circumstance which from symmetry is without bearing on the vertical compound vibrations, but becomes more marked in proportion as the vibration is twisted around into the horizontal. The latter, therefore, appears somewhat convex downward unless very long levers are chosen. The adjustment in § 24, where the circles are nearly quite perfect, is thus in many respects to be preferred, though the levers are necessarily farther apart and the lever ends incapable of resisting much tension. There is inconvenience, however, in constructing special pairs of front and rear cam axles.

To find whether the circles at the lever ends have a common cylindric envelope the cam axles should be rotated in like direction. Coincident ends should then remain nearly coincident throughout. The cross laths, *U* and *V*, are adjustable with this test in view.

40. *Rotary Polarization. Equal Component Wave-Lengths.*—Let the front cam axle be a left-handed, the rear axle a right-handed, screw (Fig. 6). Let them be equal in wave-length and amplitude. Then the component harmonics (loci of the lever ends or eyelets) will be respectively right and left circular helices, otherwise equal. The vibration lines of the particles, *W*, in Fig. 10, will all be coplanar, the plane being parallel to the cam axles at any angle to the horizontal depending on the phase difference of the initial cams. The compound harmonic, or longitudinal arrangement of the particles in the plane stated, is a simple harmonic, curve whose amplitude is the common diameter of the component circular harmonics.

This case has already been referred to in §24 and there exemplified. The compound curve, as constructed by the machine, is on a scale of one-half.

If the cam axles are rotated with the same velocity, opposite in direction (cross-belt), the corresponding plane-wave will result, unchanged in obliquity. One may note in passing that, whereas, in all the above compounding, plane-waves were obtainable in one or two special altitudes merely, they may now be obtained in all altitudes.

41. If the axles are rotated with unequal velocities, components of equal wave-length differ in period and velocity. The plane of the compound wave will, therefore, rotate about the axis of the component circles. Hence, if the oscillation of the first particle be put back into the same line after each oscillation (in general, continuously), *i. e.*, if oscillation is continually *supplied* at the origin in this line, the amount of rotation resulting will be proportional to the distances between particles. The rotary polarization so produced is due to a

difference both in the *period and the velocity* of the component circular waves of like wave-lengths.

42. *Unequal Component Wave-Lengths.*—With the front and rear cam axles still respectively left and right, if more turns be put on one than on the other, the harmonic curves will become helical. In other words, the compound of two plane simple harmonic curves of the same wave-length ratio and phase difference at the origin will now be inscribed on a regular helix. If the axles be rotated with the same angular velocity in opposite directions the component harmonics have the same period, but differ in velocity. The vibration lines in the compound wave remain fixed for each particle, but their directions differ in altitude proportionally to their distance apart. The rotary polarization so obtained is due to a difference in the *velocities* of the circular components. The helix may be rotated as a whole by dephasing the initial particles.

43. If the axles are turned with unequal velocities the helical compound wave must rotate as a whole about the common axis of the component circles, in consequence of the continuous and like dephasing at all cam pairs. Rotary polarization is again due both to difference of velocity and of period, as in § 41. If, however, the period of rotation at the cam axles is proportional to the wave-lengths of the helices the velocities of the components will be the same and the continuous rotation occurring due merely to difference in the periods of the components. Hence, if the oscillation in the first particle of the compound wave is always supplied parallel to a given line the rotary polarization obtained will be due simply to the difference in the *periods* of the components.

44. *Right-Handed Circular Component Harmonics.*—The same amount of rotation as in the last cases will be obtained when the wave-length of one of two equal right and left cam axles is increased and that of the other decreased by half the stated increase of the single axle in § 42. It will even be obtained when both cam axles are right-handed screws or both left-handed screws, alike in all respects but differing in phase by 180°, subject as before to counter rotation (cross-belted). But, whereas the rotary polarization in the preceding case, § 42, is due solely to normal advance of the circular waves, it is now due to the independent counter rotations impressed by outside agency. The two right-hand helices specified, being opposite in phase, constitute a series of stresses in equilibrium and produce no displacement.

If the cam axles of equal wave-lengths rotate with the same velocity the compound wave is a helix, but with each of its particles in the same phase. The neutral position is thus a line of balls in the common axis encircled by the lever ends, and this may be used as a test on the adjustment. Each particle persists in its line of vibration, and their locus is a helix which expands and contracts in diameter rhythmically.

45. If the two axles rotate unequally swiftly the component circular waves advance unequally swiftly and the line of vibration of each particle or the contractile helix as a whole rotates around the common axis.

46. Finally, in two right-handed cam axles of equal amplitude, but different wave-length, the resultant harmonic curve will be the compound of corresponding plane harmonics, but inscribed on the corresponding helix. For rotations of the same angular velocity (equal periods) the helical wave will not rotate as a whole. For unequal periods it will so rotate.

Some of these cases are more important than others. Their application is a question of optics.

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